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On the Weak-Noise Limit of Fokker–Planck Models

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The weak-noise limit of Fokker–Planck models leads to a set of nonlinear Hamiltonian canonical equations. We show that the existence of a nonequilibrium potential in the weak-noise limit requires the existence of whiskered tori in the Hamiltonian system and, therefore, the complete integrability of the latter. A specific model is considered, where the Hamiltonian system in the weak-noise limit is not integrable. Two different perturbative solutions are constructed: the first solution describes analytically the breakdown of the whiskered tori due to the appearance of wild separatrices; the second solution allows the analytic construction of an approximate nonequilibrium potential and an asymptotic expression for the probability density in the steady state.

KEY WORDS: Fokker–Planck processes; dynamical systems; nonequilibrium potentials; weak-noise limit; integrability of Hamiltonian systems; whiskered tori.

1. INTRODUCTION

The Fokker–Planck equation is a well-known and useful tool for investigating the dynamics of weak fluctuations in macroscopic systems.[(1–5)] Examples of its applications include not only noise in thermodynamic systems, but also nonequilibrium systems in optics, such as lasers or multistable passive optical devices or electronic systems, such as Josephson devices.

Quite often in such applications the noise is sufficiently weak that a full solution of the Fokker–Planck equation is not necessary. Instead, it is sufficient to construct asymptotic solutions, which hold in the limit of weak noise. Physical examples where this strategy was found useful are noise in

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dispersive optical bistability[6-8] or in Josephson junctions.[9] The weak-noise limit has been investigated in the mathematical literature,[10-12] too.

In the present paper we shall also be concerned with the construction of such asymptotic probability distributions. We shall restrict our attention to the probability distributions in the steady state, which are the time-independent solutions of Fokker–Planck equations. Physically, these are the most interesting solutions, since they are independent of any initial distribution and unique under very general conditions. Therefore, they reflect intrinsic properties of a system. However, mathematically, the weak-noise limit of the time-independent solutions of the Fokker–Planck equations is also the most problematical one. The reason is that really two limits are involved: the limit $t \to \infty$, in which the steady state distribution is reached from an arbitrary initial distribution and the weak-noise limit. The weak-noise limit must be carried out after the limit $t \to \infty$.

In cases of practical relevance, it is, of course, not possible to solve the Fokker–Planck equation for finite times and then to follow its solution for $t \to \infty$. Therefore, one must replace this procedure by a reasonable assumption about the form of the solution of the Fokker–Planck equation for $t \to \infty$. In order to be more specific, we introduce a formal parameter η, which characterizes the small intensity of the noise. The usual assumption for the form of the probability density $P(q,\eta)$ of the variables $q = \{q^\nu\}$, $\nu = 1, 2, \ldots, n$ in the time-independent steady state then is

$$P(q,\eta) = N(\eta) z(q) \exp\left[-\phi(q)/\eta + O(\eta) \right] \tag{1.1}$$

where $N(\eta)$ is a normalization constant and $O(\eta)$ is a correction of order η. In other words, the limits

$$\phi(q) = \lim_{\eta \to 0} \left[-\eta \ln P(q,\eta) \right] \tag{1.2}$$

$$\ln z(q) = \lim_{\eta \to 0} \left[\ln P(q,\eta) + \phi(q)/\eta - \ln N(\eta) \right] \tag{1.3}$$

are assumed to exist. In Eq. (1.2) we assumed that $\lim_{\eta \to 0} \eta \ln N(\eta) = 0$, which is satisfied in all relevant cases. The function $\phi(q)$ is often called a nonequilibrium potential of the deterministic system obtained from the Fokker–Planck process for vanishing noise.[13] Clearly, $\phi(q)$ must have a number of properties which are implied by Eq. (1.1) such as single valuedness, boundedness from below, and at least twice continuous differentiability, since (1.1) must satisfy the Fokker–Planck equation.[3] In practical applications the validity of the assumption (1.1) is usually not questioned. Instead, the success in finding at least approximate solutions for $\phi(q)$ and

[3] It appears possible that some of these conditions could be weakened, but we shall not investigate here the rather delicate questions which arise in this case.

$z(q)$ after inserting (1.1) into the Fokker–Planck equation is accepted as a check on the consistency of (1.1) with the Fokker–Planck equation.

In the present paper we want to show in a first, general part given in Section 2 that the ansatz (1.1) is, in fact, not a rigorous asymptotic solution of the Fokker–Planck equation in the generic case. The proof is given by showing that (1.1) can only apply rigorously, if a certain Hamiltonian system, which is uniquely associated with the Fokker–Planck equation, is completely integrable. The integrability of a Hamiltonian system is well known to be a nongeneric, special property. A brief account of these results has been presented previously.[14] In the light of these general results, the undeniable practical value of the ansatz (1.1) in applications may seem surprising. However, it is easily explained by the fact that many nonintegrable Hamiltonian systems show chaos only on such a small scale that it may, effectively, be ignored.

We exemplify the appearance of a nonintegrable Hamiltonian system in the weak-noise limit of the Fokker–Planck equation in Section 3, where a special model is introduced. In Section 4 we give a perturbative analysis of the Hamiltonian system corresponding to this model. The analysis shows the appearance of chaos on a scale which is exponentially small in a certain parameter ϵ of the model. In Section 5 we present an alternative perturbative analysis by expanding in ϵ. In this latter expansion the chaos in the Hamiltonian system is completely suppressed within the orders of ϵ which we considered. Using this expansion, a useful approximate expression for $P(q,\eta)$ in the form (1.1) can be calculated analytically, which we do in Section 6. In the final Section 7 we present our conclusions.

2. WEAK-NOISE LIMIT OF THE STEADY STATE DISTRIBUTION

Let us consider the Fokker–Planck equation

$$\frac{\partial P(q,\eta,t)}{\partial t} = \left[-\frac{\partial}{\partial q^\nu} K^\nu(q) + \frac{\eta}{2} Q^{\nu\mu} \frac{\partial^2}{\partial q^\nu \partial q^\mu} \right] P(q,\eta,t) \qquad (2.1)$$

with given drift $K^\nu(q)$ and diffusion coefficients $\eta Q^{\nu\mu}$. For simplicity we consider here only the case where the positive semidefinite symmetric matrix $Q^{\nu\mu}$ is independent of the variables q, but this restriction is not essential and can be removed by a straightforward generalization as long as $Q^{\nu\mu}(q)$ remains bounded. The parameter η has already been mentioned. It is introduced as a formal device to characterize the weak-noise limit $\eta \to 0$. The drift $K^\nu(q)$ and the coefficients $Q^{\nu\mu}$ are assumed to be independent of η. As boundary conditions for Eq. (2.1) we require that the probability density $P(q,\eta,t)$ and its derivatives vanish at infinity. In the following we always assume that the rather general conditions[15] are satisfied which ensure that a unique time-independent steady state solution of (2.1) is reached for $t \to \infty$ starting from any initially given probability density, and we focus our attention on this time-independent steady state probability density.

Making the ansatz (1.1) and assuming $z \neq 0$ we obtain in the weak-noise limit

$$K^{\nu}(q)\frac{\partial\phi(q)}{\partial q^{\nu}} + \frac{1}{2}\, Q^{\nu\mu}\frac{\partial\phi(q)}{\partial q^{\nu}}\,\frac{\partial\phi(q)}{\partial q^{\mu}} = 0 \tag{2.2}$$

$$\left[K^{\nu}(q) + Q^{\nu\mu}\frac{\partial\phi(q)}{\partial q^{\mu}}\right]\frac{\partial z(q)}{\partial q^{\nu}} + \left[\frac{\partial K^{\nu}(q)}{\partial q^{\nu}} + \frac{1}{2}\, Q^{\nu\mu}\frac{\partial^2\phi(q)}{\partial q^{\nu}\partial q^{\mu}}\right]z(q) = 0 \tag{2.3}$$

We note in passing that Eq. (2.2) retains its form if $Q^{\nu\mu}$ depends on q, while Eq. (2.3) is changed by additional terms containing the first derivative of $Q^{\nu\mu}$. The solution of (2.3) requires that we first solve (2.2). Therefore, we now concentrate on the solution of this equation. As was mentioned above, the ansatz (1.1) is meaningful if the function $\phi(q)$ is a single-valued, twice continuously differentiable solution of (2.2), which is bounded from below. As an additional boundary condition of the solution of (2.2) we require that $\phi(q)$ be stationary, i.e., its first derivatives vanish, in the limit sets (attractors, repellors, saddles) of the deterministic dynamical system

$$\dot{q}^{\nu} = K^{\nu}(q) \tag{2.4}$$

This condition expresses the requirement that $P(q,\eta)$ in the limit $\eta \to 0$ should have a local maximum in the attractors, a local minimum in the repellors and a saddle in the saddles of the deterministic dynamical system. Therefore, we are only interested in solutions of (2.2), which satisfy this requirement. In order to see that this requirement is compatible with Eq. (2.2) let us consider $\phi(q(t))$ as a function of time, where $q(t)$ changes according to (2.4). We then find from Eqs. (2.4), (2.2)

$$\frac{d\phi}{dt} = -\frac{1}{2}\, Q^{\nu\mu}\frac{\partial\phi}{\partial q^{\nu}}\,\frac{\partial\phi}{\partial q^{\mu}} \tag{2.5}$$

Recalling that $Q^{\nu\mu}$ is nonnegative, we observe that $\phi(q(t))$ cannot increase forward in time and cannot decrease backward in time. Since $\phi(q(t))$ is assumed to be bounded from below it cannot forever decrease in time with a finite slope, and thus $d\phi/dt$ vanishes in the limit sets. Therefore, the requirement that ϕ becomes stationary in the limit sets is automatically satisfied if $Q^{\nu\mu}$ is positive-definite, and it is at least compatible with (2.2) if $Q^{\nu\mu}$ is only positive semidefinite.

Equation (2.2) can be solved by employing the methods of characteristics. This method, in the language of classical mechanics, implies that (2.2) is interpreted as the Hamilton–Jacobi equation

$$H\left(\frac{\partial\phi(q)}{\partial q}, q\right) = 0 \tag{2.6}$$

for the action $\phi(q)$ of a system with the Hamiltonian $H(p,q)$. The comparison of Eq. (2.6) with (2.2) yields the explicit form of the Hamiltonian

$$H(p,q) = K^{\nu}(q)p_{\nu} + (1/2)\,Q^{\nu\mu}p_{\nu}p_{\mu} \tag{2.7}$$

The canonical equations following from (2.6) are

$$\begin{aligned} \dot{q}^{\nu} &= K^{\nu}(q) + Q^{\nu\mu}p_{\mu} \\ \dot{p}_{\nu} &= -\frac{\partial K^{\mu}(q)}{\partial q^{\nu}}\,p_{\mu} \end{aligned} \tag{2.8}$$

They are the equations of the characteristics of Eq. (2.2).

We are here interested only in the solutions of the time-independent Hamilton–Jacobi equation. These solutions are associated with the n-dimensional invariant manifolds of the characteristics, which, assuming for the moment there exist any, are written in the form

$$p_{\nu} = \frac{\partial\phi(q,\alpha)}{\partial q^{\nu}}, \qquad \nu = 1, 2, \ldots, n \tag{2.9}$$

The parameters α_{λ}, $\lambda = 1, 2, \ldots, n$, are constants of integration, one of which, e.g., α_1, is given by the value of the Hamiltonian (2.7), which vanishes according to Eq. (2.6):

$$\alpha_1 = H(p,q) \equiv 0 \tag{2.10}$$

The remaining $n-1$ parameters $\alpha_2, \ldots, \alpha_n$ are needed to parametrize the invariant manifolds of Eqs. (2.8). If the Hamiltonian system (2.8) is completely integrable, the invariant manifolds (2.9) give a smooth foliation of phase space by n-dimensional hypersurfaces, and Eqs. (2.9) solved for $\alpha_1, \ldots, \alpha_n$ define n smooth phase space functions, which are called constants of the motion. If the Hamiltonian system is not completely integrable, at least one of these phase-space functions is not smooth, i.e., it is impossible to find n constants of the motion. For instance in a system with 2 variables, $n = 2$, the Hamiltonian (2.7) would be the only smooth phase-space function if (2.8) is not completely integrable.

In order to extract from Eq. (2.9) the desired function $\phi(q)$, we must choose the parameters α_{λ} in (2.9) in such a way that

$$\frac{\partial\phi(q,\alpha)}{\partial q^{\nu}} = 0, \qquad \nu = 1, \ldots, n \tag{2.11}$$

in the limit sets of the deterministic equation (2.4). We denote the union of the limit sets of (2.4) by Γ. We now want to analyze under which conditions (2.11) can be satisfied on Γ by an appropriate choice of the α_λ.

First of all, let us note that a trivial n-dimensional invariant manifold S_0 of (2.8) always exists and is given by

$$S_0\colon \quad p_\nu = 0, \qquad \nu = 1, \ldots, n \tag{2.12}$$

By (2.9) it corresponds to the trivial solution $\phi(q) = \text{const}$ of Eq. (2.2). On S_0 the Hamiltonian dynamics (2.8) reduces to (2.4). Therefore, all the limit sets of (2.4) are also limit sets of (2.8). Of course, the opposite need not be true. All parts of Γ are connected by the invariant manifold S_0 of (2.8). However, there must exist additional n-dimensional invariant manifolds of (2.8) transverse to S_0 which emanate from the different parts of Γ. These additional invariant manifolds, of course, exist only in the Hamiltonian system (2.8) and have no meaning in the deterministic equations (2.4) which are restricted to S_0.

Let us now consider a particular limit set Ω in Γ. Writing the n-dimensional invariant manifold of (2.8) transverse to S_0 emanating from this limit set Ω in the form (2.9), we define locally, in the vicinity of this limit set, a function $\phi_\Omega(q)$, which, by construction, satisfies Eq. (2.11) on the particular limit set Ω which we have chosen. We see, therefore, that it is always possible to choose the parameters α_λ in (2.9) in such a way that (2.11) is at least satisfied on any single given limit set Ω of Eq. (2.4). However, in order to obtain an acceptable solution of Eq. (2.2) we need much more: we need that it be possible to choose the parameters α_λ in such a way that (2.11) is satisfied at once everywhere on Γ. In other words, the n-dimensional invariant manifolds which emanate transverse to S_0 from the different limit sets which make up Γ must be local parts of a single, globally defined manifold, $p_\nu = \partial\phi(q)/\partial q^\nu$, $\nu = 1, 2, \ldots, n$, with $\phi(q) \to \phi_\Omega(q)$ for $q \to \Omega$. We conclude that in order to obtain an acceptable solution of Eq. (2.2) the Hamiltonian system (2.8) must admit smooth n-dimensional separatrices $p_\nu = \partial\phi(q)/\partial q^\nu$, $\nu = 1, 2, \ldots, n$, which join the different parts of Γ in phase space. As we mentioned before, the n-dimensional invariant manifold S_0 also connects the different parts of Γ. Therefore, S_0 and the smooth separatrices must form closed n-dimensional surfaces or "whiskered tori," which are a landmark of completely integrable Hamiltonian systems.[16,17]

Our considerations can, therefore, be summarized as follows. A necessary condition for the existence of a single-valued, twice continuously differentiable solution of Eq. (2.6), which satisfies (2.9) on Γ and is bounded from below is the complete integrability of the Hamiltonian system (2.8) at $H \equiv 0$. We believe that this necessary condition is actually quite strong. However, it may not be a sufficient condition in all cases, e.g., due to the condition of single valuedness and boundedness from below. It is conceivable that one could construct completely integrable Hamiltonians of the form (2.7) for which these latter conditions are violated, even though we are not aware of such examples.

We want to recall at this point that the integrability of the Hamiltonian system (2.8) has also another important consequence, which was discussed in a previous paper,[14] namely, the existence of a potential of the deterministic drift $K^\nu(q)$. By this we mean the possibility to express $K^\nu(q)$ in terms of a $\phi(q)$ by

$$K^\nu(q) = -\frac{1}{2} Q^{\nu\mu} \frac{\partial \phi}{\partial q^\mu} + r^\nu(q) \tag{2.13}$$

where the part $r^\nu(q)$ of $K^\nu(q)$, which cannot be derived from $\phi(q)$ is a drift on equipotential surfaces and satisfies

$$r^\nu(q) \frac{\partial \phi}{\partial q^\nu} = 0 \tag{2.14}$$

It is well known that complete integrability is a very special, nongeneric property of Hamiltonian systems. Therefore, the result which we have just derived has the immediate consequence that a solution $\phi(q)$ of Eq. (2.2) with the desired properties will usually not exist. This may seem surprising in view of the fact that at least to our knowledge, no single example has been discussed in the literature, where the ansatz (1.1) was found to fail. We have found, however, that at least for 2-variable systems, $n = 2$, it is very easy to construct such examples using the method of Poincaré cross sections (see, e.g., Ref. 17), which reduces the dynamics (2.8) to a two-dimensional area-preserving map. The following sections are, therefore, devoted to the study of a class of concrete examples, which exemplify the general results of this section.

3. THE MODEL

We consider the Fokker-Planck dynamics of an overdamped anharmonic oscillator with a periodic forcing term described by a drift $K^\nu(x, y)$, where

$$\begin{aligned} K^1(x, y) &= \epsilon\big(x - x^3 + f(x)\cos y\big) \\ K^2(x, y) &= \omega = \text{const} \end{aligned} \tag{3.1}$$

and a diffusion coefficient $Q = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in (2.1). The latter means that in a Langevin description a Gaussian white noise term would appear in the x equation only. We choose the time unit in such a way that $\omega = 1$. The function $f(x)$ will be represented by its Taylor series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{3.2}$$

The variable y is a phase and the x, y space is the cylinder $-\infty < x < \infty$, $0 \leqslant y < 2\pi$.

It follows from the general considerations (Section 2) that the Hamiltonian associated with the dynamical system is given as

$$H = p_x^2/2 + p_x \epsilon\big(x - x^3 + f(x)\cos y\big) + p_y \tag{3.3}$$

Without periodic perturbations ($f(x) \equiv 0$) y is a cyclic variable and the system is integrable. For $H \equiv 0$ the solution in this case reads

$$\begin{aligned} p_x &= -\epsilon(x - x^3) \pm \left[\epsilon^2(x - x^3)^2 - 2p_y\right]^{1/2} \\ p_y &= \alpha_2 = \text{const} \end{aligned} \tag{3.4}$$

The manifolds (3.4) give a smooth foliation of the three-dimensional energy hypersurface $H \equiv 0$. In the x, p_x plane we obtain smooth curves, which are independent of y (Fig. 1). In particular, for small positive values of α_2 the cross sections of closed invariant tori appear in the region between the x axis and the parabola $p_x^{(0)} = -2\epsilon(x - x^3)$ for $|x| < 1$. Integrating the equa-

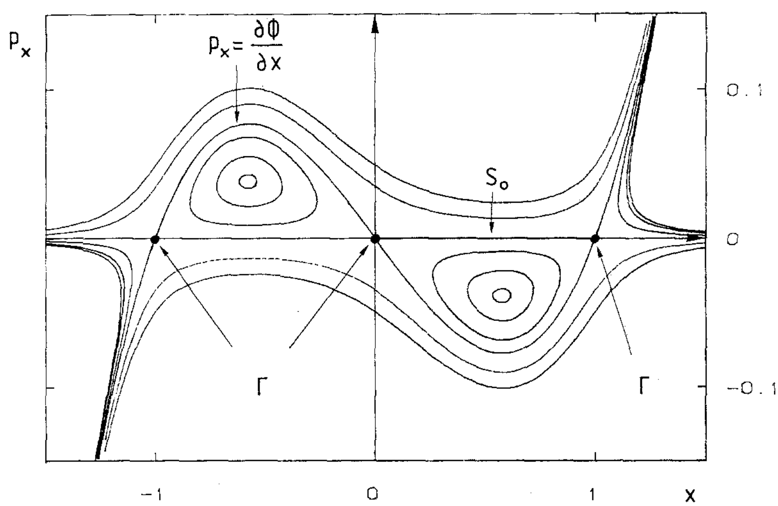

Fig. 1. Invariant curves of the x, p_x plane of the Hamiltonian system (3.7) in the integrable case, $f(x) \equiv 0$ ($\epsilon = 0.1$). Γ and S_0 denote the limit sets and the hyperplane $p_x = p_y = 0$, respectively.

tions

$$p_x = \frac{\partial \phi}{\partial x}, \qquad p_y = \frac{\partial \phi}{\partial y} \tag{3.5}$$

one may construct the solutions $\phi(x, y, \alpha_2)$ of the corresponding Hamilton–Jacobi equation. Among the different possibilities, indexed by α_2, there is, however, only one solution which can be considered as a potential of the system. Only

$$\phi \equiv \epsilon\phi_0 = -\epsilon(x^2 - x^4/2) \tag{3.6}$$

fulfills the requirement of single valuedness and twice continuous differentiability, of boundedness from below and the requirement that the two-dimensional manifold $p_x = \epsilon\partial\phi_0/\partial x$, $p_y = \epsilon\partial\phi_0/\partial y$ passes through the limit sets of the deterministic system (2.4) (i.e., the sets $\{x = 0,\ 0 \leqslant y < 2\pi\}$ and $\{x = \pm 1,\ 0 \leqslant y < 2\pi\}$). The x axis and the curve $\epsilon\partial\phi_0/\partial x$ represent "whiskered tori" among the limit set points.

In the presence of a periodic perturbation the Hamiltonian system is no longer integrable, the whiskered tori cease to exist. As we shall see, if one finds a solution $p_x = \partial\phi/\partial x$, $p_y = \partial\phi/\partial y$ which passes through one of the limit sets, it will not pass through the others. Wild separatrices appear in the system and certain trajectories become chaotic. We have numerically solved the Hamiltonian equations of (3.3):

$$\begin{aligned}
\dot{x} &= \epsilon\big(x - x^3 + f(x)\cos y\big) + p_x \\
\dot{y} &= 1 \\
\dot{p}_x &= -\epsilon p_x\big(1 - 3x^2 + (df(x)/dx)\cos y\big) \\
\dot{p}_y &= \epsilon p_x f(x)\sin y
\end{aligned} \tag{3.7}$$

at $H \equiv 0$ for different choices of $f(x)$ [e.g., $f(x) = \text{const}, x, x^3, x^5, x - x^3$] and investigated the Poincaré cross section (x, p_x) at $y = 0$. We have found in all cases that in the vicinity of the limit set points the motion is irregular. Figure 2 shows the Poincaré cross section near the origin for $f(x) = x - x^3$. The picture is characteristic of nonintegrable Hamiltonian systems: regimes of regular and irregular motion can be distinguished. The borderline of the region investigated is the unstable separatrix emanating from $P_+ = (1,0)$. [For $f(x) = a(x - x^3)$ not only $P_0 = (0,0)$ but $P_\pm = (\pm 1, 0)$ remain limit set points, too.]

As long as ϵ is small, the chaotic motion occupies tiny neighborhoods of the limit sets only. We shall see that apart from these regions the separatrices can be very well approximated by smooth analytic functions, which render it possible to define an approximate potential for the dissipa-

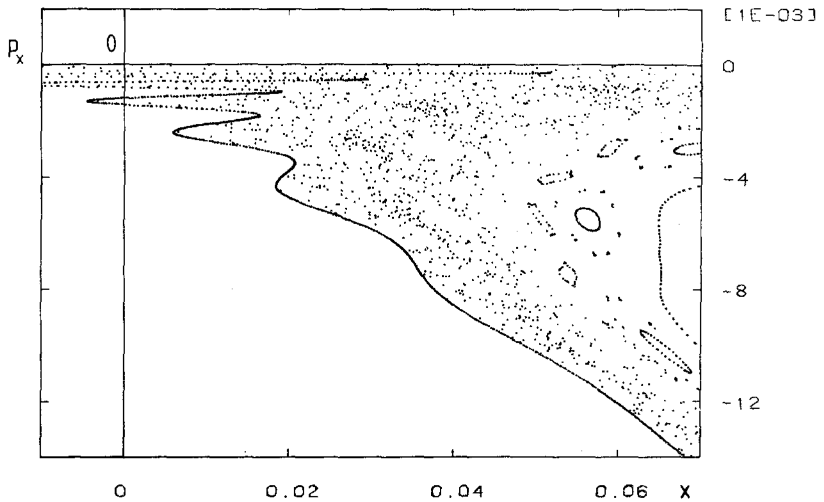

Fig. 2. The Poincaré surface (x, p_x plane at $y = 0$) for $f(x) = x - x^3$, $\epsilon = 0.1$ near the origin. The dots belong to the trajectories with the initial conditions $x_0 = 0.05, 0.1, \ldots, 0.65$, $p_{x0} = -\epsilon x_0$. The borderline of this region is the wild separatrix emanating from the limit set point $(1, 0)$.

tive system. In the next section we shall discuss two complementary expansion methods which help to understand the behavior of the system inside and outside the dominantly chaotic region. Approximate potentials of some dissipative models with a single attracting fixed point have been constructed along similar lines in Ref. 18.

4. THE EXPANSION IN f

The first perturbative method we consider is an expansion in powers of $f(x)$. The Taylor coefficients a_n, and thus $f(x)$ itself, are assumed to be small in the region of interest. At the same time ϵ is arbitrary.

In a first order calculation we can write $\phi(x, y)$ as

$$\phi(x, y) = \epsilon\phi_0(x) + W_1(x, y) \tag{4.1}$$

where $\epsilon\phi_0$ is given by (3.6) and W_1 is proportional to the small parameters of $f(x)$. The equation specifying W_1 follows from the Hamilton–Jacobi equation to be

$$\frac{\partial W_1}{\partial x} - \frac{1}{\epsilon(x - x^3)} \frac{\partial W_1}{\partial y} = -2\epsilon f(x)\cos y \tag{4.2}$$

The characteristics of the homogeneous part of Eq. (4.2) are given by

$$y(x) = -\frac{1}{2\epsilon}\ln\frac{x^2}{1-x^2} + C \tag{4.3}$$

therefore, the general solution of the homogeneous equation can be formally written as

$$W_{1h} = I\left(e^{-iy}(1-x^2)^{i/(2\epsilon)}x^{-i/\epsilon}\right) \tag{4.4}$$

where I is an arbitrary function.

When looking for a particular solution of the inhomogeneous equation we keep only one Taylor coefficient, $a_n x^n$, of $f(x)$ and replace $\cos y$ by $\exp(-iy)$. At the end we take the real part of the solution obtained in this way and add all the contributions of the different Taylor components.

Integrating the equation of W_1 along the characteristics we obtain

$$W_{1p} = -2a_n\epsilon^2 \mathrm{Re}\left[\frac{x^{n+1}(1-x^2)^{i/(2\epsilon)}e^{-iy}}{i+(n+1)\epsilon} \times F\left(\frac{n+1}{2}+\frac{i}{2\epsilon}, \frac{i}{2\epsilon}, \frac{n+3}{2}+\frac{i}{2\epsilon}, x^2\right)\right] \tag{4.5}$$

as a particular solution, where F stands for the hypergeometric function. From the general solution $W_1 = W_{1h} + W_{1p}$ the momentum p_x can be easily calculated by means of (4.2) to be

$$p_x = \frac{\partial\phi}{\partial x} = -2\epsilon(x-x^3) + \frac{1}{\epsilon(x-x^3)}\frac{\partial W_1}{\partial y} - 2a_n\epsilon x^n \cos y \tag{4.6}$$

First we consider the case $n \neq 0$ and determine the solution p_x which satisfies the condition $p_x(0, y) = 0$. This solution is associated with the stable manifold of the origin of the x, p_x plane. The joint requirement of $p_x(0, y) = 0$ and 2π periodicity in y uniquely fix the arbitrary function I in Eq. (4.4) as a constant. The function p_x is, therefore, uniquely determined and smooth near $x = 0$. The same p_x, however, starts to oscillate near $x = \pm 1$, i.e., the separatrix does not remain smooth in a first-order calculation already. We may rewrite W_1 using the transformation formulas of the hypergeometric functions[19] and obtain

$$W_1 = -2a_n\epsilon^2(1-x^2)x^{n+1}\left[\cos y\,\mathrm{Re}\frac{F_n(x^2)}{i+(n+1)\epsilon} + \sin y\,\mathrm{Im}\frac{F_n(x^2)}{i+(n+1)\epsilon}\right] \tag{4.7}$$

where

$$F_n(x^2) \equiv F\left(1, \frac{n+3}{2}, \frac{n+3}{2} + \frac{i}{2\epsilon}, x^2\right) \tag{4.8}$$

Consequently,

$$p_x = -2\epsilon(x - x^3) - 2a_n \epsilon x^n \left\{ -\sin y \operatorname{Re} \frac{F_n(x^2)}{i + (n+1)\epsilon} + \cos y \left[1 + \operatorname{Im} \frac{F_n(x^2)}{i + (n+1)\epsilon} \right] \right\} \tag{4.9}$$

Considering now the Poincaré surface $y = 0$ and taking the limit $|x| \to 1$, the asymptotic behavior of p_x turns out to be given by

$$p_x^s = -a_n \frac{\pi \operatorname{sgn}^n(x) R_n}{\Gamma((n+3)/2) 2\epsilon \sinh(\pi/2\epsilon)} \frac{1}{1 - x^2} \times \sin\left[\frac{1}{2\epsilon} \ln(1 - x^2) + \arctan \delta_n\right] \tag{4.10}$$

where

$$R_n e^{i \arctan \delta_n} = \frac{\Gamma((n+1)/2 + i/2\epsilon)}{\Gamma(1 + i/2\epsilon)} \tag{4.11}$$

Hence, p_x exhibits oscillations with increasing amplitude as $|x|$ tends to 1. Note, however, that the region where the wild behavior dominates is exponentially small for small values of ϵ, and the oscillatory part of p_x has a nonanalytic dependence on ϵ. Therefore, if an ϵ expansion is performed, it does not include the nonanalytic terms and provides an approximate smooth separatrix, as well as a corresponding approximate potential (Section 5).

Instead of requiring $p_x(0, y) = 0$ we may require $p_x(1, y) = 0$ [or, alternatively, $p_x(-1, y) = 0$]. In that case the unspecified function $I(u)$ in W_{1h} (4.4) must be chosen as linear in u. After using again the transformation formulas of the hypergeometric functions we obtain

$$W_1 = -2a_n \epsilon^2 (1 - x^2) x^{n+1} \left[\cos y \operatorname{Re} \frac{F_n(x^2)}{i - 2\epsilon} + \sin y \operatorname{Im} \frac{F_n(x^2)}{i - 2\epsilon} \right] \tag{4.12}$$

where

$$F_n(x^2) \equiv F\left(1, \frac{n+3}{2}, 2 - \frac{i}{2\epsilon}, 1 - x^2\right) \tag{4.13}$$

The corresponding momentum now reads

$$p_x = -2\epsilon(x - x^3) - 2a_n\epsilon x^n\left\{-\sin y \operatorname{Re}\frac{F_n(x^2)}{i-2\epsilon} + \cos y\left[1 + \operatorname{Im}\frac{F_n(x^2)}{i-2\epsilon}\right]\right\} \quad (4.14)$$

This expression, however, diverges at $x = 0$. Its asymptotic behavior on the Poincaré surface for $x \to 0$ is given by

$$p_x^s = -a_n \frac{\pi R_n}{\Gamma((n+3)/2)2\epsilon \sinh(\pi/2\epsilon)} \frac{1}{x} \sin\left(\frac{1}{\epsilon}\ln x - \arctan \delta_n\right) \quad (4.15)$$

where R_n and δ_n have been defined by (4.11). In the vicinity of $|x| = 1$ p_x (4.14) is a smooth function of x. At $x = 1$ it has a small value, $-8a_n\epsilon^3/(1+4\epsilon^2)$, on the Poincaré surface. As the slope of the unperturbed solution at $|x| = 1$ is 4ϵ, we can interpret the above result by saying that p_x (4.14) describes the unstable manifold of the new limit set points situated now at

$$x_+ = 1 + a_n 2\epsilon^2/(1+4\epsilon^2), \qquad x_- = -1 + a_n(-1)^n 2\epsilon^2/(1+4\epsilon^2) \quad (4.16)$$

on the x axis. As we have seen, these separatrices exhibit oscillations with increasing amplitudes in the vicinity of the origin. (It is to be noted that for $n = 0$ the third limit set point does not remain in the origin, its shift is proportional to a_0.)

In the limit of small values of ϵ it is easy to estimate the width of the region around the limit set points where the wild behavior dominates by equating p_x^s to the unperturbed result. We obtain

$$\Delta x = \left\{\frac{a_n \pi \exp[-\pi/(2\epsilon)]}{\Gamma((n+3)/2)(2\epsilon)^{(n+3)/2}}\right\}^{1/2} \quad (4.17)$$

This result shows that for modest values of n the size of the macroscopically chaotic regions is increasing with n in accordance with the numerical findings.

The perturbative method presented here loses its validity where the corrections to $p_x^{(0)}$ are of the same order of magnitude as $p_x^{(0)}$ itself, i.e., just within a region of size Δx around the limit set points. Up to the borderline of this region, however, the calculated curves approximate very well the oscillating separatrices. Figure 3 shows the separatrix emanating from the limit set point $(1,0)$ in the vicinity of the origin and the result of the f expansion (dashed line) for $f(x) = a(x - x^3)$, $a = \epsilon = 0.1$. The continuous line represents the stable separatrix of the origin approximated in this

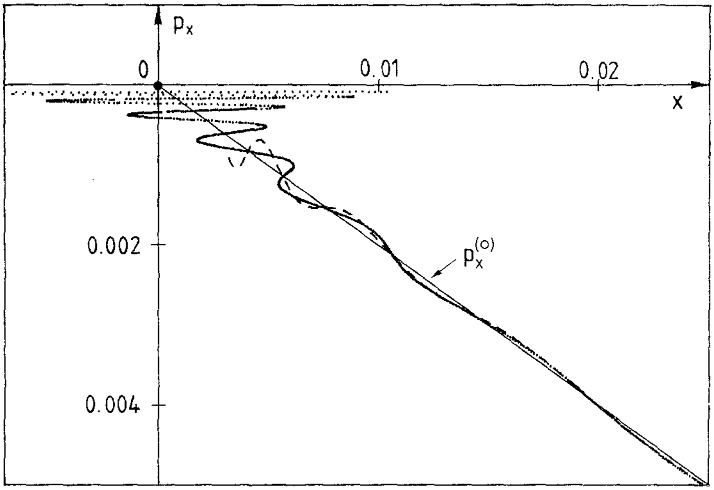

Fig. 3. The wild separatrix of the limit set point (1, 0) near the origin of the Poincaré surface for $f(x) = 0.1(x - x^3)$, $\epsilon = 0.1$. The dashed line is the analytic approximation obtained in the f expansion in first order. The full line represents $p_x^{(0)} = -2\epsilon(x - x^3)$ which agrees in this region with the momentum $p_x = \partial\bar{\phi}/\partial x$ derived from the approximate potential.

regime by $p_x^{(0)} = -2\epsilon(x - x^3)$. Heteroclinic points can be noticed. It is remarkable that the heteroclinic points generated by the curve we calculated approximately agree well with the real ones even far inside the chaotic region where the curve itself is no longer a good approximation. One can easily see that the x coordinates of the heteroclinic points generated by the intersection of p_x [Eq. (4.14)] and $p_x^{(0)}$ form a geometric series for small values of ϵ with the quotient $\exp(-\epsilon\pi)$.

The simple example we considered here illustrates the general property of Fokker–Planck models in the weak-noise limit that systems with a stationary distribution in the form (1.1) are special. Already a small perturbation of such special systems causes qualitative changes as it makes the associated Hamiltonian system in general nonintegrable. If one finds a separatrix emanating from one limit set, it does not pass through the other limit set and, therefore, the construction of a potential $\phi(q)$ with the required properties is not possible.

On the other hand, one may use the wild separatrices in regions where they are smooth to construct very useful local approximations of $P(q, \eta)$ of the form (1.1). In particular, if we use the wild separatrix (4.14) which emanates smoothly from the limit set $|x| = 1$ and make use of Eqs. (4.12) and (4.1) we find to first order in the parameter a a local approximation for $\phi(x, y)$ near the attractors of the deterministic equations, where the probability density (1.1) is large. This local approximation only breaks down in a small vicinity of the repellor at $x = 0$, which is a region where the probability density (1.1) is in any case very small.

5. THE ϵ EXPANSION

We now turn to the discussion of another perturbative method which provides results as power series in ϵ and, therefore, does not reflect the nonanalytic dependence on this variable. As the wild oscillations of the separatrices are related to nonanalytic terms in ϵ, the present expansion is expected to give a good approximation for the nonoscillating component of the separatrices. Since, however, for small values of ϵ this component dominates apart from a tiny neighborhood of the limit set points, such an expansion turns out to be a powerful tool for calculating approximate separatrices and approximate potentials for dynamical systems.

Owing to the simplicity of the method, we can apply it for a more general system than our original one given by (3.1). We consider the Hamilton–Jacobi equation

$$\frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{\partial \phi}{\partial x}\epsilon(g(x) + f(x)\cos y) + \frac{\partial \phi}{\partial y} = 0 \tag{5.1}$$

where $g(x)$ and $f(x)$ are arbitrary functions, and ϕ is known to be 2π periodic in y. The leading order solution is proportional to ϵ, thus, we may look for a solution in the form

$$\phi(x, y) \approx \bar{\phi}(x, y) = \sum_{n=0}^{\infty} \epsilon^{n+1}\phi_n(x, y) \tag{5.2}$$

Here $\bar{\phi}(x, y)$ is assumed to differ from a solution of Eq. (5.1) only by terms which are exponentially small in ϵ. The calculation of $\phi(x, y)$ will be performed here up to third order in ϵ, which is already sufficient to illustrate the main ideas. Substituting (5.2) in (5.1) we find in order ϵ that $\partial\phi_0/\partial y = 0$. In second order one obtains

$$\frac{1}{2}\left(\frac{\partial \phi_0}{\partial x}\right)^2 + \frac{\partial \phi_0}{\partial x} g(x) + \frac{\partial \phi_0}{\partial x} f(x)\cos y + \frac{\partial \phi_1}{\partial y} = 0 \tag{5.3}$$

Since ϕ_0 depends on x only, and ϕ_1 cannot contain a term proportional to y as this would mean aperiodicity in y, the sum of the first two terms of (5.3) must vanish separately, leading to the results

$$\phi_0 = -2\int g(x)\,dx \tag{5.4}$$

$$\phi_1 = 2g(x)f(x)\sin y + G_1(x) \tag{5.5}$$

$G_1(x)$ is to be specified by the requirement of periodicity of $\bar{\phi}$. In the next order

$$\frac{\partial \phi_2}{\partial y} = \frac{\partial \phi_1}{\partial x}\left[g(x) - f(x)\cos y \right] \tag{5.6}$$

is found, thus ϕ_2 may be periodic in y only if $G_1(x)$ is a constant. Integrating then (5.6) over y, one obtains

$$\phi_2 = \left[g(x) f(x) \right]' \left[f(x)\cos 2y - 4g(x)\cos y \right]/2 + G_2(x) \tag{5.7}$$

where the prime denotes derivation. Finally, the periodicity of ϕ_3 fixes G_2 to be

$$G_2(x) = f(x)\left[f'(x)g(x) - f(x)g'(x) \right] - 2\int g(x)f(x)f''(x)\,dx \tag{5.8}$$

As an illustrative example, we consider the case $f(x) = ag(x) = a(x - x^3)$ which belongs to the family of models which has been investigated in the previous section. Up to third order in ϵ we obtain

$$\begin{aligned}\bar{\phi}(x, y) = &-\epsilon(x^2 - x^4/2) + 2\epsilon^2 a x^2(1 - x^2)^2 \sin y \\ &+ \epsilon^3\Big[a x^2(1 - x^2)^2(1 - 3x^2)(a\cos 2y - 4\cos y) \\ &+ a^2 x^4(3 - 4x^2 + 3x^4/2)\Big]\end{aligned} \tag{5.9}$$

while the corresponding momentum on the Poincaré surface $y = 0$ reads

$$\bar{p}_x = -2\epsilon(x - x^3)\left\{ 1 + \epsilon^2\left[4a(1 - 9x^2 + 12x^4) - a^2(1 - 3x^2 + 6x^4) \right] \right\} \tag{5.10}$$

This approximate separatrix passes through all the limit set points $(-1, 0)$, $(0, 0), (1, 0)$. In a vicinity of these points, which is exponentially small in ϵ, the approximate separatrix $\bar{p}_x$ qualitatively differs from the separatrix obtained in the numerical simulation or that found by means of the previous expansion method (Fig. 3). For small values of ϵ, however, $\bar{p}_x$ provides a very good approximation in the intermediate regions. This is illustrated on Fig. 4 which shows the numerically calculated separatrices and $\bar{p}_x$ in an intermediate region. Here, the separatrices coincide within the numerical accuracy, and they are quite well approximated by $\bar{p}_x$. It is worth mentioning that the precision of the numerical determination of the separatrices can be improved by using the calculated approximate separatrix $\bar{p}_x$ near the emanating point rather than $p_x^{(0)} = -2\epsilon(x - x^3)$.

Finally, we note that up to the linear terms in the parameter a $\bar{\phi}(x, y)$ coincides with the results obtained from Eq. (4.1) by expanding both potentials W_1 calculated in the previous Section [i.e., (4.7) and (4.12)]. This fact supports the view that the oscillating parts of the wild separatrices are owing to nonanalytic terms.

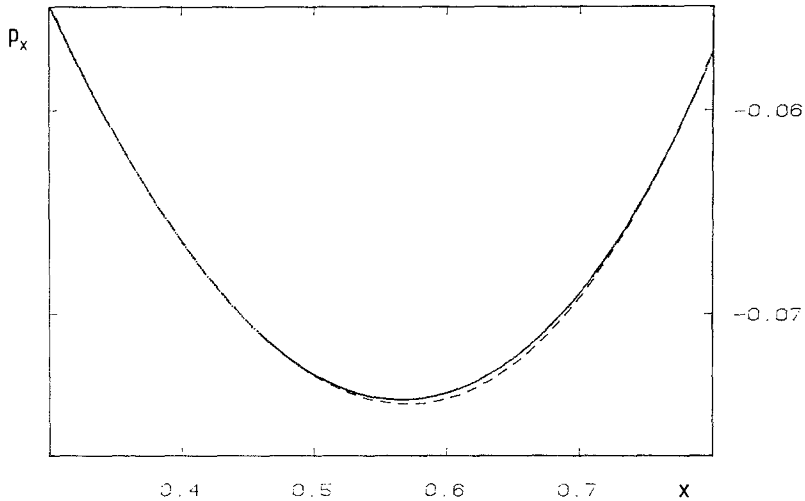

Fig. 4. The separatrices of (0, 0) and (1, 0) in a region away from the limit set points. In the region shown, both separatrices coincide within numerical accuracy. The dashed line is the result of the ϵ expansion $[f(x) = x - x^3, \epsilon = 0.1]$.

6. THE APPROXIMATE PROBABILITY DISTRIBUTION

After having found the approximate potential $\bar{\phi}(x, y)$ we now turn to the calculation of the approximate probability distribution. By substituting

$$P(x, y) \propto z(x, y)\exp\left[-\bar{\phi}(x, y)/\eta \right] \tag{6.1}$$

into the Fokker–Planck equation (2.1) with K^ν given by (3.1) and $Q^{\nu\mu}$ defined after (3.1) one obtains for $z(x, y)$

$$\frac{\partial z}{\partial x}\left(\epsilon g + \epsilon f \cos y + \frac{\partial \bar{\phi}}{\partial x}\right) + \frac{\partial z}{\partial y} + \left(\epsilon g' + \epsilon f' \cos y + \frac{1}{2}\frac{\partial^2 \bar{\phi}}{\partial x^2}\right) z = 0 \tag{6.2}$$

which is Eq. (2.3) specialized to the present example.

The ϵ expansion of this equation can be worked out along similar lines as that of (5.1) after $\bar{\phi}$ has been determined in powers of ϵ. We may now write

$$z(x, y) = \sum_{n=0}^{\infty} \epsilon^n z_n(x, y) \tag{6.3}$$

It then follows from the leading order term that z_0 may depend on x only. Using (5.4), the terms proportional to ϵ give

$$\frac{\partial z_1}{\partial y} = z_0'(x)(g(x) - f(x)\cos y) - z_0(x) f'(x)\cos y \tag{6.4}$$

As the periodicity of z_1 in y requires z_0 to be constant, we find

$$z_1(x, y) = -z_0 f'(x)\sin y + h_1(x) \tag{6.5}$$

Proceeding in a similar way up to order ϵ^2 one finds that the arbitrary function $h_1(x)$ of Eq. (6.5) must be a constant and one obtains, apart from a multiplicative constant, for the prefactor

$$\begin{aligned} z(x, y) = 1 &- \epsilon f'(x)\sin y \\ &+ \epsilon^2\{[g''(x)f(x) + 2g'(x)f'(x) + 2g(x)f''(x)]\cos y \\ &\quad + [2f(x)f''(x) - f'^2(x)]/4 \\ &\quad - [f(x)f''(x) + f'^2(x)](\cos 2y)/4\} \end{aligned} \tag{6.6}$$

We evaluate this expression again for $f(x) = ag(x) = a(x - x^3)$ to find

$$\begin{aligned} z(x, y) = 1 &- \epsilon a(1 - 3x^2)\sin y \\ &+ \epsilon^2[a(2 - 30x^2 + 36x^4)\cos y \\ &\quad - a^2(1 - 12x^2 + 15x^4)(\cos 2y)/4 - a^2(6x^2 - 3x^4 + 1)/4] \end{aligned} \tag{6.7}$$

Figure 5 shows the corresponding probability distribution (6.1) at $a = 1$,

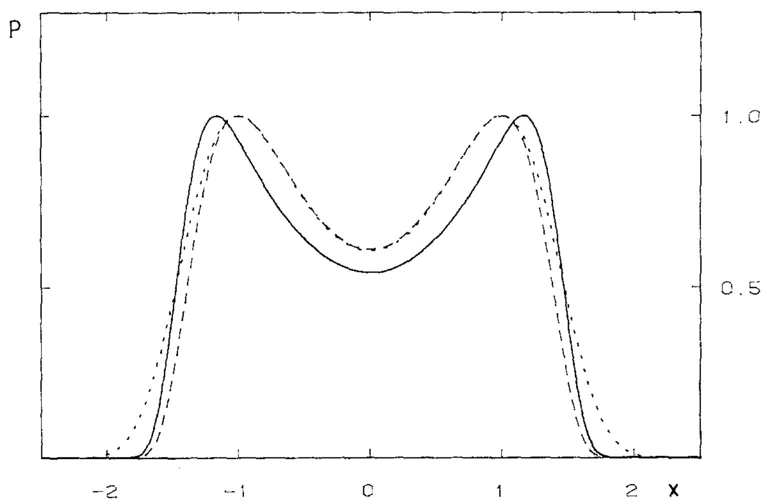

Fig. 5. The approximate probability distribution (6.1) for $\epsilon = \eta = 0.1$, $a = 1$, $y = 0$ normalized in such a way that the maximum values are unity. The distribution without the prefactor is given by the dashed line, that corresponding to the integrable case ($a = 0$) is given by the dotted line.

$\epsilon = \eta = 0.1$ for $y = 0$. In order to see the influence of the prefactor z, the distribution $P \propto \exp(-\bar{\phi}/\eta)$ and the distribution for the integrable case $f \equiv 0$ are also shown. It is to be noted that, owing to the prefactor z, the most probable states are shifted away from $x = \pm 1$. This shows that for small but finite η the most probable states are close to but not identical with the deterministic attractors.

7. CONCLUDING REMARKS

We have shown in this paper that Fokker–Planck models with a unique steady state probability density may be subdivided into two classes which are distinguished by their behavior in the weak-noise limit.

In the first class, which is rather special, fall all Fokker–Planck models for which the associated Hamiltonian (2.7) is completely integrable. As we have shown, the first class contains all Fokker–Planck models whose steady state probability density is of the form (1.1) in the weak-noise limit.

In the second class fall all Fokker–Planck models with nonintegrable Hamiltonian (2.7). Therefore, this class is much broader than the first one. The models in this class do not have a steady state probability density of the form (1.1) with the required properties in the weak-noise limit.

Important members of the first class are Fokker–Planck models describing fluctuations in thermodynamic equilibrium. Here, the existence of a solution $\phi(q)$ of the Hamilton–Jacobi equation (2.6) is ensured by the potential conditions.[20] According to these conditions $K^{\nu}(q)$ and $Q^{\nu\mu}$ may be written as

$$K^{\nu}(q) = -\frac{1}{2} Q^{\nu\mu} \frac{\partial \psi(q)}{\partial q^{\mu}} + r^{\nu}(q) \tag{7.1}$$

where $\psi(q)$ is a thermodynamic potential and where the first part on the right-hand side transforms like q^{ν} under the microscopically defined transformation of time reversal, while the second part, $r^{\nu}(q)$, transforms like $\dot{q}^{\nu}$. $\psi(q)$ and $r^{\nu}(q)$ satisfy, in addition to (7.1),

$$\eta \frac{\partial r^{\nu}(q)}{\partial q^{\nu}} - r^{\nu}(q) \frac{\partial \psi(q)}{\partial q^{\nu}} = 0 \tag{7.2}$$

From Eqs. (7.1), (7.2) it follows that $\psi(q)$ is a solution of the Hamilton–Jacobi equation in the limit $\eta \to 0$.

Fokker–Planck models of nonequilibrium systems will usually fall into the second class. Exceptions are one-variable systems with natural boundary conditions at infinity, and linear Gaussian systems, described by Ornstein–Uhlenbeck processes. For these systems the Hamiltonian (2.7) is always integrable, i.e., they belong to the first class.

Even though most nonequilibrium systems will not have a steady state probability density of the form (1.1) in the weak-noise limit, Eq. (1.1) still remains useful as an approximate ansatz. Furthermore, by a judicious choice of the solution it is possible, in many examples, to confine the regions where the solution breaks down to the vicinity of repellors or saddles of the deterministic system, where the probability density of the stochastic system is, in any case, small. In order to illustrate these points we have discussed specific models in Sections 3–6. They provide explicit examples of nonintegrable Hamiltonians of the form (2.7) and they illustrate the usefulness of the ansatz (1.1) for obtaining approximate densities in the steady state.

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